More properties of homomorphisms

If $\phi : G \to H$ is a homomorphism then:

1a) $\phi(e_G) = e_H$ 1c) $\forall g \in G,\ n \in \mathbb{Z},\ \phi(g^n) = \phi(g)^n$

2) $\phi(G) \leq H$ 3) $\ker(\phi) = \{g \in G : \phi(g) = e_H\} \leq G$

4) If $\phi : G \to H$ is an isomorphism, then so is $\phi^{-1} : H \to G$.

(already proved)

5a) If $g \in G$ and $|g| < \infty$ then $|\phi(g)| \mid |g|$.

ϕ is a hom.

Pf: Suppose $|g| = n \in \mathbb{N}$. Then $e_H \underset{1a}{=} \phi(e_G) = \phi(g^n) = \phi(g)^n$

$\Longrightarrow |\phi(g)| \mid n$. $\blacksquare$

Lemma from video about cyclic groups

5b) If ϕ is an isomorphism then $\forall g \in G$, $|g| = |\phi(g)|$.

Pf: First suppose that $|g| < \infty$. Then by 5a, $|\phi(g)| \mid |g|$,

so $|\phi(g)| < \infty$. By 4, ϕ^{-1} is an isom., so

$$\Rightarrow |g| \mid |\phi(g)|$$

But then $|g| = |\phi(g)|$.

Next, we want to show that if $|g| = \infty$ then $|\phi(g)| = \infty$.

Equivalently, if $|\phi(g)| < \infty$ then $|g| < \infty$. (contrapositive)

So suppose $|\phi(g)| < \infty$. Since ϕ^{-1} is an isom.,

$$|g| = |\phi^{-1}(\phi(g))| \mid |\phi(g)| \Rightarrow |g| < \infty. \quad \blacksquare$$

5c) If ϕ is an isomorphism then $\forall n \in \mathbb{N}$,

$$\#\{g \in G : |g| = n\} = \#\{h \in H : |h| = n\}.$$

Pf: It follows from 5b that ϕ is a bijection from the set on the left to the set on the right. $\blacksquare$

Exs: 1) The groups C_8, $C_2 \times C_4$, and $C_2 \times C_2 \times C_2$ are pairwise non-isomorphic.

- $C_8 = \langle x \mid x^8 = e \rangle$, $\quad |x| = 8$
- $C_2 \times C_4 = \langle y_1 \mid y_1^2 = e \rangle \times \langle y_2 \mid y_2^4 = e \rangle = \{ (y_1^i, y_2^j) : 0 \le i \le 1,\ 0 \le j \le 3 \}$
  - $(y_1^i, y_2^j)^4 = ((y_1^2)^{2i}, (y_2^4)^j) = (e,e) \Rightarrow |(y_1^i, y_2^j)| \mid 4$.
  - $|(e, y_2)| \mid 4$, $(e, y_2)^1 \neq (e,e)$, $(e, y_2)^2 = (e, y_2^2) \neq (e,e) \Rightarrow |(e, y_2)| = 4$.
- $C_2 \times C_2 \times C_2 = \langle z_1 \mid z_1^2 = e \rangle \times \langle z_2 \mid z_2^2 = e \rangle \times \langle z_3 \mid z_3^2 = e \rangle$

  $= \{ (z_1^i, z_2^j, z_3^k) : 0 \le i, j, k \le 1 \}$
  - $(z_1^i, z_2^j, z_3^k)^2 = ((z_1^2)^i, (z_2^2)^j, (z_3^2)^k) = (e,e,e) \Rightarrow |(z_1^i, z_2^j, z_3^k)| \mid 2$.
- C_8 has an element of order 8, but neither $C_2 \times C_4$ nor $C_2 \times C_2 \times C_2$ does. Therefore $C_8 \not\cong C_2 \times C_4$ and $C_8 \not\cong C_2 \times C_2 \times C_2$.
- $C_2 \times C_4$ has an element of order 4, but $C_2 \times C_2 \times C_2$ does not, so $C_2 \times C_4 \not\cong C_2 \times C_2 \times C_2$.

2) $D_8 \not\cong Q_8$

- $D_8 = \langle r, s \mid r^4 = s^2 = e,\ rs = sr^{-1} \rangle = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$|e| = 1$, $|r| = 4$, $|r^2| = 2$, $|r^3| = 4$, $|s| = 2$, $|sr| = 2$,

$\left(\begin{aligned} (sr)(sr) &= s(rs)r \\ &= s(sr^{-1})r = e \end{aligned} \right)$

$|sr^2| = 2$, $|sr^3| = 2$.

- $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$(i^2 = -1,\ i^4 = 1)$

$|1| = 1$, $|-1| = 2$, $|i| = 4$, $|-i| = 4$, $|j| = 4$, $|-j| = 4$,

$|k| = 4$, $|-k| = 4$.

Q_8 has 6 elements of order 4, but D_8 has 2, so $D_8 \not\cong Q_8$.

6) If ϕ is an isomorphism then G is Abelian if and only if H is Abelian.

Pf: Suppose G is Abelian. Let $h_1, h_2 \in H$ and choose $g_1, g_2 \in G$ with

$\phi(g_1) = h_1$, $\phi(g_2) = h_2$. Then

$$h_1 h_2 = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1) = h_2 h_1.$$

Therefore H is Abelian.

On the other hand, if H is Abelian, we can use the fact that ϕ^{-1} is an isom to deduce, in the same way as above, that G is Abelian. $\blacksquare$

Ex. 3) The groups C_8, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, D_8, and Q_8 are pairwise non-isomorphic.

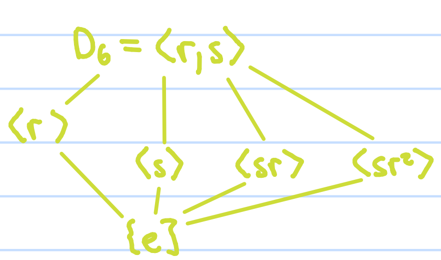
Follows from exs. 1 & 2 and the fact that C_8, $C_2 \times C_4$, and $C_2 \times C_2 \times C_2$ are Abelian, but D_8 and Q_8 are not.

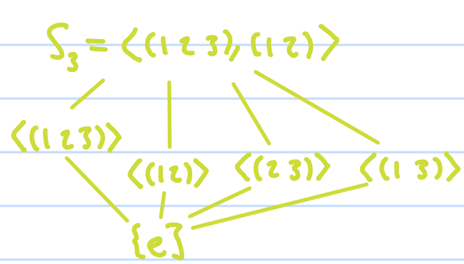
7) If ϕ is an isomorphism then G and H "have the same lattices of subgroups." i.e. there is a bijective correspondence between subgroups of G and subgroups of H, which preserves orders of subgroups and subgroup inclusions.

Warning: The converse of this is not true in general. There are examples of non-isomorphic groups which have the same lattices (even taking into account orders of subgroups).

Exs:

4) $D_6 \cong S_3$ (mentioned before ... will prove soon)





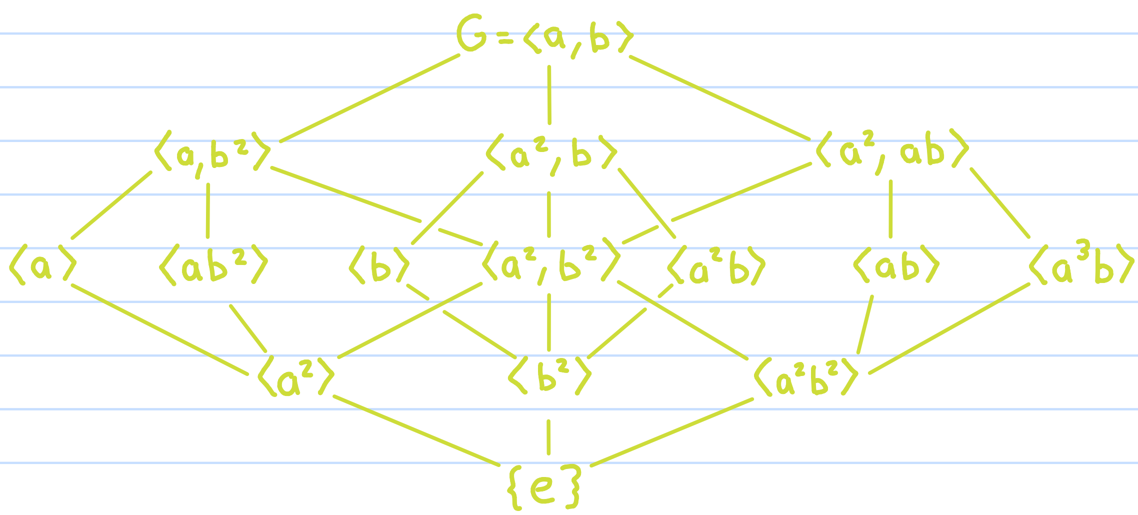
5) Example related to warning:

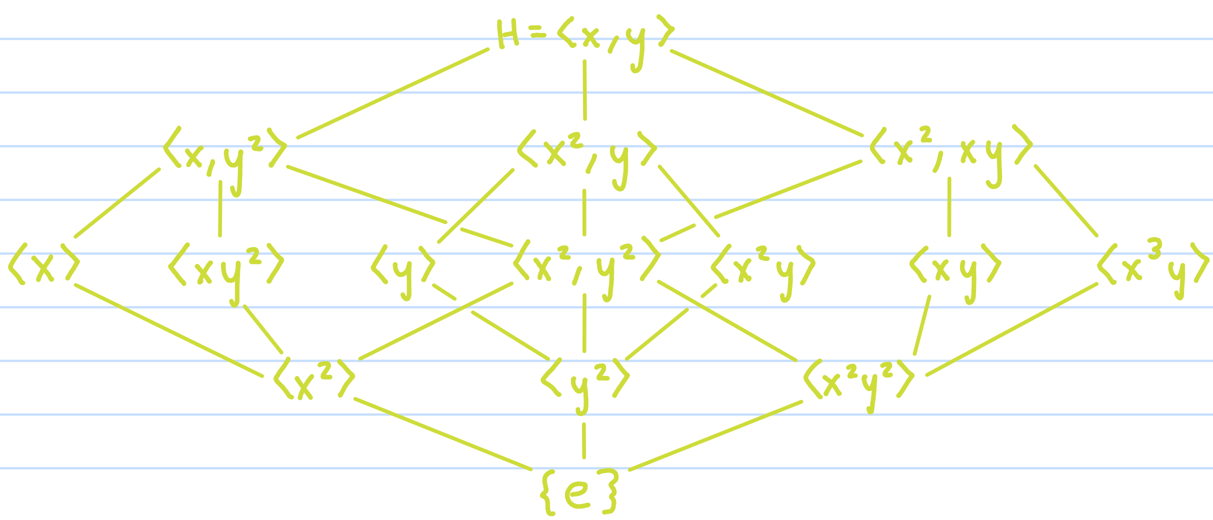
Let $G = C_4 \times C_4 = \langle a, b \mid a^4 = b^4 = e,\ ab = ba \rangle$,

and $H = C_4 \rtimes C_4 = \langle x, y \mid x^4 = y^4 = e,\ xy = y^{-1}x \rangle$.

("semi-direct product")

Then G is Abelian but H is not, so $G \not\cong H$. However, they have the same number of subgroups of each order, with the same lattice of subgroups. (no pair of groups with orders ≤ 15 has this property)





Note also: All corresponding pairs of proper subgroups in these lattices are isomorphic.